Skew-symmetric elements in nonlinear involutions in group rings

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Abstract
Given an involution in a group $G$, it can be extended in various ways to an involution in the group ring $RG$, where $R$ is a ring, not necessarily commutative. In this paper nonlinear extensions are considered and necessary and sufficient conditions are given on the group $G$, its involution, the ring $R$ and the extension for the set of skew-symmetric elements to be commutative and for it to be anticommutative.

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1 Introduction
Group rings have been studied in the last years from the point of view of rings with an involution $\ast$. Within this trend, it is customary to study the set of symmetric elements, those fixed by the involution, $\alpha^\ast = \alpha$, and the set of skew-symmetric elements, those for which $\alpha^\ast = -\alpha$, for they usually impose restriction on the structure of the whole ring. The main properties studied within these sets are commutativity and anticommutativity, which share many similar results.

Commutativity was the first to attract attention. Bovdi, Kovács and Sehgal [1] studied the so called natural involution in the group ring $RG$ of a group $G$ and a commutative ring $R$, which is the $R$-linear extension of the involution $g \mapsto g^{-1}$ in the group. They where interested in the set of symmetric units, which form a subgroup of the set of units of $RG$ only if the former is commutative. They gave the answer for $G$ a locally finite $p$-group and Bovdi and Parmenter [2] got it for integral group rings of periodic groups. Then Bovdi [3], in addition to widen the answer, made the link between the symmetric units and the symmetric elements: in case $R$ is a $G$-favourable integral domain, the former being a subgroup is
equivalent to the latter being a subring. And the set of symmetric elements being a subring is equivalent to it being commutative. Therefore, Broche attacked directly this question: under which conditions is the set of symmetric elements of $RG$ a commutative set? He gave the answer in [4].

After it, Jespers and Ruiz Marín considered then a generalization by taking an arbitrary involution $*$ in the non-Abelian group $G$ which was to be extended $R$-linearly to the group ring $RG$. Again, the ring $R$ is asked for to be commutative for this construction to work. They gave a complete answer in [15].

A further generalization in the same line has been recently proposed by García-Delgado and Raposo in [9]. Now the involution $*$ in the group is arbitrary as well, but they consider any extension of it to an involution in the group ring $RG$. Now the ring $R$ needs to be commutative no more. In their paper they give necessary and sufficient conditions in the group $G$, the involution $*$, the ring $R$ and the extension of the involution to $RG$ so as the set of symmetric elements is commutative.

Commutativity has also been studied in the set of skew-symmetric elements. Broche and Polcino Milies, in [6], gave necessary and sufficient conditions on the group $G$ so as the set of skew-symmetric elements of the natural involution in the group ring $RG$ is a commutative set. Jespers and Ruiz Marín, together with the former authors, in a series of two papers, [14, 7], studied the same question but for an arbitrary involution in the group, extended $R$-linearly to the group ring. In both cases, the ring needs to be commutative.

On the other hand, anticommutativity has also attracted attention. This is a property more suited to the set of skew-symmetric elements than to the set of symmetric elements because the identity in $R$ is of the latter class and, if anticommutativity holds, then $1 = -1$ and, thus, the characteristic of the ring is 2. But in this case anticommutativity is the same as commutativity, so the problem is already solved. In addition, anticommutativity of the set of skew-symmetric elements is equivalent to this set being a subring (just as commutativity works for the symmetric elements). Goodaire and Polcino Milies have, thus, studied anticommutativity of the set of skew-symmetric elements in the scenario of an arbitrary involution in the group $G$ extended $R$-linearly to the group ring $RG$.

The present paper follows the line opened by García-Delgado and Raposo of an involution in $RG$ which is an extension of an involution in $G$, but not necessarily $R$-linear, and I focus now on the set of skew-symmetric elements. I study both question within this set: commutativity and anticommutativity.

Another kind of involution which has been considered within group rings is worth mentioning: oriented involutions. The same kind of questions have been posed and answered for this involution: symmetric elements in [5] and [13] and skew-symmetric elements in [8] and [13]. But this involution, though is a generalization of the $R$-linear extension of an arbitrary involution in $G$, is not a extension as such, so it is not included in the present work.

Thus I start with a given involution $*$ in the group $G$, and I extend it to $*: RG \rightarrow RG$, an involution in the group ring (i.e. $*$ restricted to $G$ is the original involution), where $R$ is a ring with identity, not necessarily commutative. If I
apply the involution $*$ to an element $a$ of $R$ its image is $a^* = \sum_{g \in G} b_g g$ but, after applying it again $a = a^{**} = \sum_{g \in G} b_g^* g^*$. Since $*$ is an involution in $G$, I must conclude $b_1^* = a$ and $b_g^* = 0$ and, thus, $b_g = 0$, for $g \neq 1$. This amounts to say that the map $*$, when restricted to the ring $R$, is an involution in this last ring. Notice that the linear extension is recovered if this involution in $R$ is the identity function. Also notice that, in char $R = 2$ the sets of symmetric elements and skew-symmetric elements coincide and the former is already solved in the paper by García-Delgado and Raposo, so I assume throughout this paper char $R \neq 2$.

I write $R G^-$ and $R^-$ for the sets of skew-symmetric elements of $R G$ and $R$ respectively and $G^+$ for the set of symmetric elements of $G$. The additive commutator in a ring is denoted $[a, b] = ab - ba$, while the Jordan product is denoted $a \circ b = ab + ba$. Also $Z(R)$ and $Z(G)$ denote the centres of $R$ and $G$ respectively.

With this notation, the goal of this paper can be stated as giving necessary and sufficient conditions on the group $G$, the involution in $G$, the ring $R$ and the involution in $R$ so as the set $R G^-$ is commutative, in section 2, or anticommutative, in section 3.

Consider a skew-symmetric element $\alpha = \sum_{g \in G} a_g g$ in $R G$. It is convenient to split its image, $\alpha^*$, into two parts:

$$\alpha^* = \sum_{g \in G^+} a_g^* g + \sum_{g \in G \setminus G^+} a_g^* g^*,$$

(1)

where the second summation can be equivalently written as $\sum_{g \in G \setminus G^+} (a_g^*)^* g$. The condition $\alpha^* = -\alpha$ leads then to

$$\sum_{g \in G^+} (a_g + a_g^*) g + \sum_{g \in G \setminus G^+} (a_g + (a_g^*)^*) g = 0.$$

(2)

It is clear thus, on the one hand, that $a_g$ must be a skew-symmetric element of $R$ whenever $g$ is in $G^+$. On the other hand, $a_g^* = -a_g^-$ if $g \in G \setminus G^+$. This last equality allows to reorganize the second summation by joining together the term with element $g$ and the term with element $g^*$. Therefore, any skew-symmetric element $\alpha$ may be written as

$$\alpha = \sum_{g \in G^+} a_g g + \sum_{g \in L} (a_g g - a_g^* g^*),$$

(3)

where $a_g \in R^-$ for $g \in G^+$. Here the set $L$, is built from $G \setminus G^+$ by taking one element from each couple $\{g, g^*\}$. This can be summarized by saying that the set of skew-symmetric elements in $R G$ is additively generated by the set $R \cup S$, where $R = \{a g : a \in R^-, g \in G^+\}$ and $S = \{a g - a^* g^* : a \in R, g \in G \setminus G^+\}$.

2 Commutativity of skew-symmetric elements

Commutativity in $R G^-$ is assured if the set $R \cup S$ is commutative, so the proofs in this section deal with commutators between elements from $R$, between
elements from $S$ and between an element from $R$ and an element from $S$.

Because the problem of $RG^-$ being commutative is already solved when the involution on $R$ is the identity in $[14, 7]$ I focus on the case when this involution is not the identity function on $R$. This condition implies $R^-$, the set of skew-symmetric elements of $R$, is not trivial, for if $a$ is an element in $R$ for which $a^* \neq a$, then the skew-trace $a - a^*$ is a nonzero skew-symmetric element.

Being $R^-$ a subset of $RG^-$, it is clear that the former is commutative whenever the latter is, but the first lemma, which follows, goes further, as it says that the former set is in the centre of the ring.

**Lemma 2.1.** If the involution in the group $G$ is not the identity and $RG^-$ is commutative, then $R^-$ is contained in the centre of $R$.

*Proof.* Let $a$ be in $R^-$, $b$ an arbitrary element of $R$ and $h$ an element in $G \setminus G^+$. Then both $a$ and $bh - b^*h^*$ are in $RG^-$, so they commute by the hypothesis. Expanding the commutator in $[a, bh - b^*h^*] = 0$, I get $(ab - ba)h + (b^*a - ab^*)h^* = 0$. Since $h \neq h^*$, both coefficients must be zero so, in particular, $ab = ba$. \hfill \Box

**Lemma 2.2.** If $RG^-$ is commutative then either $G^+$ is a commutative set or $(R^-)^2 = 0$.

*Proof.* Assume there are skew-symmetric elements $a$ and $b$ in $R$ such that $ab \neq 0$. Take $g$ and $h$ elements in $G^+$. Then $ag$ and $bh$ commute for both are skew elements of $RG$, so $ab gh = ba hg = ab hg$ because $R^-$ is commutative. Therefore, $gh = hg$ as stated. \hfill \Box

In the following lemmas I get some more details of the involution in the group $G$, but I need to distinguish if it is or it is not an Abelian group. In case $G$ is an Abelian group, then $R$ being a commutative ring renders $RG$ also a commutative ring, so the question about $RG^-$ is trivial. This is why next lemma, which is useful in the case of $G$ Abelian, asks for a noncommutative ring.

**Lemma 2.3.** Let $R$ be a noncommutative ring and $G$ a group in which there are elements $g$ and $h$ which are not symmetric and $gh = hg$. If $RG^-$ is commutative, then the product $gh$ is in $G^+$ and every additive commutator in $R$ is a symmetric element. In particular, if $G$ is an Abelian group, then $G^+$ is a subgroup of index 2 in $G$, there is an element $s$ in $G^+$ of order 2 and the involution is given by $g^* = g$ or $sg$.

*Proof.* By the hypothesis $[g - g^*, h - h^*] = 0$ and, since $gh = hg$, also $g^*h^* = h^*g^*$, so this leads to

$$gh^* + g^*h = h^*g + hg^*.$$ (4)

If $gh^*$ were different from $h^*g$ and from $hg^*$ then it would be $gh^* = g^*h$ and the characteristic of the ring 2. Because this scenario has been discarded from the beginning, there are but two choices left: (i) $gh^* = hg^*$ and also $g^*h = h^*g$ or (ii) $gh^* = h^*g$.
Now let $a$ and $b$ be elements in $R$ such that $ab \neq ba$. By the hypothesis $[ag - a^*g^*, bh - b^*h^*] = 0$. Expanding the commutator, and assuming case (ii) in which $gh^* = h^*g$, I get

$$[a, b] gh + [a^*, b^*] g^*h^* = [a, b^*] gh^* + [a^*, b] g^*h.$$  \hfill (5)

Clearly $gh$ is not in the support of the right side, so neither in that of the left. Therefore, since $[a, b] \neq 0$ it must be $gh = g^*h^*$, which amounts to say $(gh)^* = gh$, and $[a, b] + [a^*, b^*] = 0$, which can be written as $[a, b^*] = [a, b]$. Of course this last equation is trivially satisfied by commuting elements of $R$, so it is valid with any commutator. The case (i) is managed similarly and gives the same result.

In case $G$ is an Abelian group, the condition $gh \in G^+$ for any elements $g$ and $h$ in $G \setminus G^+$ suffices to prove $G^+$ is a subgroup of index 2 in $G$ with an element $s$ of order 2 such that $g^* = g$ or $sg$, as is showed by Goodaire and Polcino Milies in [12].

Next two lemmas are intended for non-Abelian groups.

**Lemma 2.4.** Assume there are elements $g$ in $G^+$ and $h$ in $G \setminus G^+$ with $gh \neq hg$ and $*$ restricted to $R$ is not the identity. If $RG^-$ is commutative and the involution in $R$ is not the identity function, then $gh = g^*h^*$ and $(T_R)^2 = 0$, i.e. the product of traces in $R$ is trivial.

**Proof.** Let $a$ be a nonzero skew element in $R$, $b$ an arbitrary element of the same ring and $g$ and $h$ as stated in the lemma. The elements $ag$ and $bh - b^*h^*$ are both skew-symmetric, so they commute, from which

$$ab gh + ab^* h^*g = ab hg + ab^* gh^*,$$  \hfill (6)

where the result of lemma 2.1, $R^-$ is in the centre, has been taken into account. Since $gh \neq hg$ and $gh \neq gh^*$, $gh$ is not in the support of the right side of the equation, so it is neither on the left side. If $ab \neq 0$ (which is true, for instance, if $b = 1$) the only way for equation (6) to hold is $gh = h^*g$ and $ab + ab^* = 0$. Meanwhile, on the right side, $hg = gh^*$, so I get $h^* = g^{-1}hg = ghg^{-1}$ and $a(b^* + b^*) = 0$ as demanded.

In case there is $b$ such that $ab = 0$, then it is also true that $ab^* = 0$, so $a(b^* + b^*) = 0$ as well. \hfill \Box

**Lemma 2.5.** Let $g$ and $h$ be in $G \setminus G^+$ with $gh \neq hg$. If $RG^-$ is commutative and the involution in $R$ is not the identity function, then $gh = h^*g = g^*h$ and $(T_R)^2 = 0$, i.e. the product of traces in $R$ is trivial.

Notice the result of the set of traces having trivial product also contains the result that char $R = 4$, since 2 is a trace and, thus, 4=0 while char $R \neq 2$ by hypothesis.

**Proof.** The proof of this lemma has two parts. The first one relies heavily on lemma 1.1 of [14] which states that, under the hypothesis of the present
lemma, elements $g$ and $h$, and the ring $R$, must verify one of the following seven conditions:

1. $gh \in G^+$, $hg \in G^+$, $hg^* = gh^*$ and $h^*g = g^*h$.
2. $gh \in G^+$, $hg \in G^+$, $hg^* = g^*h$.
3. $gh \in G^+$, $hg = g^*h = gh^*$ and char $R = 3$.
4. $gh = hg^* = g^*h^*$, $h^*g = g^*h$ and char $R = 3$.
5. $gh = h^*g = g^*h^*$, $hg^* = gh^*$ and char $R = 3$.
6. $hg \in G^+$, $gh = h^*g = hg^*$ and char $R = 3$.
7. $gh = hg^* = h^*g = g^*h^*$ and char $R = 3$.

I now follow these seven choices in the analysis of the commutator $[a(g + g^*), h - h^*]$, where $a$ is a nonzero skew element in $R$. This commutator is zero because both are skew elements, so I get

$$a(gh + g^*h + h^*g + h^*g^*) = a(gh^* + g^*h^* + hg + hg^*).$$

(7)

1. In the first of the seven choices, substitution into equation (7) gives $2a(gh + g^*h) = 2a(gh + gh^*)$. Since $gh \neq hg$, $gh \neq gh^*$ and $gh \neq g^*h$ I conclude it necessary $2a = 0$ for the equation to hold.

2. In lemma 1.2 of [14], together with lemma 3.3 of [7] it is proved that this case implies case 1, so the result is the same.

3. Conditions in item 3 carry equation (7) into $2a gh + a h^*g = a hg + 2a h^*g$ which implies $gh = h^*g$. But, taking into account we are under the assumption $gh = h^*g^*$, this leads to $h^*g = h^*g^*$ and, thus, to the contradiction $g = g^*$. So this case is not possible.

4. Here equation (7) gets the form $2a g^*h = a(gh + hg)$, which is clearly a contradiction because $gh \neq hg$, so this case is neither possible.

Cases 5 and 6 lead, respectively to equations $2a gh^* = a(gh + hg)$ and $2a hg = a(gh + gh^*)$, which are both contradictions as well.

Therefore, from the six different cases (as the first two are the same) only two of them survive, the first and the last ones.

Now, in the second part, the first choice is also discarded. Consider arbitrary elements $a$ and $b$ in the ring $R$ and the elements $g$ and $h$ mentioned in the lemma. The commutator $[ag - a^*g^*, bh - b^*h^*]$ is zero because the elements are skew-symmetric. Its expansion under the first of the choices aforementioned gives

$$(ab - b^*a^*)gh + (ba^* - ab^*)gh^* = (ba - a^*b^*)hg + (a^*b - b^*a)h^*g.$$

(8)

By the hypothesis $gh$ does not equal $gh^*$, nor equals $hg$ nor even equals $h^*g$ because, in the latter $gh = h^*g = g^*h$ and, so, $g = g^*$, a contradiction. Hence its coefficient must be zero, i.e., $b^*a^* = ab$, which is the same as $(ab)^* = ab$ for arbitrary $a$ and $b$ in the ring $R$. In particular, for $b = 1$, I get $a^* = a$, so the involution in $R$ is the identity, a contradiction with the hypothesis. So this option is not possible.

Only the option numbered 7, which reads $gh = hg^* = h^*g = g^*h^*$, is a valid one. Under it, the expansion of the commutator gives

$$(ab + b^*a + ba^* + a^*b)gh = (ba + ab^* + a^*b + b^*a)hg.$$

(9)
Since $gh \neq hg$, I get
\[ ab + b^*a + ba^* + a^*b^* = 0, \] (10)
which contains the result $\text{char } R = 4$ by taking $a = b = 1$. Also, taking $b = 1$ in equation (10) I get $2(a + a^*) = 0$ so $T_R$, the set of traces in $R$, is 2-torsion and, whence, traces are also skew-symmetric elements. Now, since skew elements in $R$ are in the centre, $a$ and $b - b^*$ commute, which yields the identity
\[ ab + b^*a = ba + ab^*. \] (11)
Upon substitution of (11) into equation (10) I get $b(a + a^*) + (a + a^*)b^* = 0$. But, due to $T_R$ being in the centre of $R$ (because is a set of skew-symmetric elements), this can be written as $(a + a^*)(b + b^*) = 0$.

Lemmas 2.4 and 2.5 show that, in a non-Abelian group, if $g^* \neq g$ and $gh \neq hg$ then $g^* = h^{-1}gh = hgh^{-1}$. Goodaire and Polcino Milies have shown in [12] (theorem 2.2) this condition is sufficient to characterize such a group as having a unique nontrivial commutator, $s$ (necessarily central and of order 2), and its involution being given by $g^* = g$ or $sg$. It is worthy to depart briefly on these groups with such an involution. Their quotient by the centre is noticeable because it is an elementary Abelian group. A subfamily is distinguished: those groups in which the set of symmetric elements, $G^+$, is commutative (and, thus, a subgroup). In this case, $G^+$ coincides with the centre of the group and $G$ satisfies an additional property denoted as LC (after lack of commutativity or also limited commutativity) which says that two elements commute if, and only if, any of them is in the centre or their product is in the centre of the group. This also means that elements of different cosets, relative to the centre, do not commute, which is possible only if the quotient with the centre collapses to the simplest form: $G/Z(G) \approx C_2 \times C_2$, where $C_2$ is the cyclic group of order 2. These groups within the family of groups with a single nontrivial commutator are referred to as SLC groups and are extensively studied in [11].

Next theorem shows the necessary conditions exhibited so far turn out to be sufficient. It gives the answer to the problem of $R^-$ being commutative in case the extension of $*$ to $RG$ is nonlinear.

**Theorem 2.6.** Let $G$ be a group with involution $*$, $R$ a ring of characteristic different from 2 and $RG$ the associated group ring in which an involution, which is an extension of $*$, has been defined. If this extension is nonlinear, the set of skew-symmetric elements, $R^-$, is commutative if, and only if, one of the following scenarios is found:

i. $G$ is an Abelian group and $R$ is a commutative ring.

ii. $G$ is an Abelian group with the identity function as involution and the extension is such that $R^-$ is commutative.

iii. $G$ is an Abelian group with a subgroup $H$ of index 2, an element $s$ in $H$ of order 2 and the involution given by $g^* = g$, if $g$ is in $H$, and $g = sg$ if $g$ is not in $H$. 

7
The involution in $R$, which is a noncommutative ring, verifies $R^-$ is in the centre of $R$ and every additive commutator is a symmetric element.

iv. $G$ is a non-Abelian group with a unique nontrivial commutator, $s$, and the involution is given by $g^* = g$ or $sg$.

The involution in $R$ satisfies $R^-$ is in the centre of $R$, every additive commutator is a symmetric element and, if $G^+$ is commutative, $(T_R)^2 = 0$, while if it is not, $(R^-)^2 = (R^-)(T_R) = 0$.

Proof. It is clear that (i) or (ii) are both sufficient conditions. If the group is Abelian, its involution is not the identity and the ring is not commutative, then lemmas 2.1 and 2.3 say the conditions in (iii) are necessary. Now I show they are also sufficient. An element from the set $\mathcal{R}$ of the form $ag$, with $a \in R^-$, $g \in G^+$. Since $R^-$ in in the centre of $R$ and $G$ is Abelian, it clearly commutes with any other element of $RG$. Therefore I have to deal only with commutators between elements of $S$ but, if $g$ and $h$ are in $G \setminus G^+$ and $a$ and $b$ are arbitrary elements of the ring $R$, then $[ag - a^*g^*, bh - b^*h^*] = ([a, b] - [a, b^*])gh - ([a, b^*] - [a, b^*])sgh$, which is zero by the hypothesis on commutators.

Now assume $G$ is a non-Abelian group. Lemmas 2.1, 2.4 and 2.5 say it is necessary $G$ has a unique nontrivial commutator, $s$, and the involution is given by $g^* = g$ or $sg$, while $R^-$ is in the centre of the ring $R$ and every commutator is a symmetric element. In addition, if $G^+$ is commutative, and so $G \setminus G^+$ is not, lemma 2.5 imposes the condition $(T_R)^2 = 0$ too, while if $G^+$ is not commutative, then lemmas 2.2 and 2.4 demand $(R^-)^2 = (R^-)(T_R) = 0$.

Now I prove these conditions are also sufficient, so let $G$ be a non-Abelian group with a unique nontrivial commutator $s$ and an involution given by $g^* = g$ or $sg$, where the set $G^+$ is commutative, and $R$ a ring with involution such that $R^- \subset Z(R)$ and $(T_R)^2 = 0$. Then $G^+$ is in the centre of the group and, thus, $ag$ with $a \in R^-$, $g \in G^+$ commute with any element in $RG$ so the elements in $R$ commute among them and with those in $S$. Now, with regard to the set $S$, consider elements $g$ and $h$ in $G \setminus G^+$ and $a$ and $b$ in $R$. The commutator $[ag - a^*g^*, bh - b^*h^*]$ expands, if $gh = hg$, as $([a, b] - [a, b^*])gh + ([a, b^*] - [a, b^*])sgh$, which is zero because of the hypothesis on the commutators. However, if $gh \neq hg$, the commutator expands as $(ab + b^*a + a^*b + a^*b^*)g(h - (ba + ab^* + a^*b + b^*a^*))h$.

Now it suffices to prove the first parenthesis is zero, as the second is the $s$-image of the first. Because of the nilpotency of the set of traces $(a + a^*)(b + b^*) = 0$, which gives the equation

$$ab + ab^* + a^*b + a^*b^* = 0.$$  

First take $a = b = 1$ in equation (12), from where I get, since $\text{char } R \neq 2$, $\text{char } R = 4$. Now take $b = 1$ in equation (12), then $2(a + a^*) = 0$, so $T_R$ is a 2-torsion set, which means traces are skew as well as symmetric elements and, therefore, they are in the centre of the ring. Hence, the commutator $[a + a^*, b]$ is zero and I get the equation

$$ab + a^*b = ba + b^*.$$

(13)
On the other hand, the commutator \([b - b^*, a]\) is also zero since \(b - b^*\) is skew-symmetric and, whence, central. It gives the equation

\[ ba + ab^* = ab + b^*a . \]  

(14)

The addition of equations (13) and (14) gives

\[ a^*b + ab^* = b^*a + ba^* \]  

(15)

which, upon substitution in equation (12) leads to

\[ ab + b^*a + ba^* + a^*b^* = 0 . \]  

(16)

Now, if \( G \setminus G^+ \) is commutative and, thus, \( G^+ \) is not so, then the commutator between \( ag \) and \( bh \), with \( a, b \) in \( R \) and \( g, h \) in \( G^+ \) is \([ag, bh] = abgh - ba hg\) which is zero because the product in \( R^-\) is trivial. The commutator \([ag, bh - b^*h^*]\), where \( a \in R^-\), \( b \in R \), \( g \in G^+ \) and \( h \in G \setminus G^+ \) is expanded as \( abgh - ab hg - ab^* gh^* + ab^* h^*g \) because \( R^-\) is in the centre. If \( gh = hg \) this is zero and, if it is not, then \( gh^* = sgh = hg \) and \( h^*g = shg = gh \) and thus the commutator is \( a(b + b^*)(gh - hg) \) which is zero because the product of a skew element and a trace is zero by hypothesis. Finally, the commutator \([ag - a^*g^*, bh - b^*h^*]\), with \( a, b \in R \) and \( g, h \in G \setminus G^+ \) is zero by the same arguments as in the first paragraph of this proof, referred to the case of the group \( G \) Abelian, which apply here as well.

\[ \square \]

### 3 Anticommutativity of skew-symmetric elements

Anticommutativity in the set of skew-symmetric elements can be expressed as the Jordan product being trivial, \( \alpha \circ \beta = \alpha \beta + \beta \alpha = 0 \) for \( \alpha \) and \( \beta \) skew-symmetric elements. As in the preceding section, this amounts to assure the set \( R \cup S \) defined in the introduction is anticommutative.

The first result in this direction is a lemma of Goodaire and Polcino Milies [12] which fixes the characteristic of the ring and which I repeat here for ease of further reference.

**Lemma 3.1** (lemma 2.1 of [12]). Assume the involution in \( G \) is not the identity. If \( RG^-\) is anticommutative then \( \text{char} \ R = 4 \), \( gg^* = g^*g \) and \((g^2)^* = g^2 \) for all \( g \) in \( G \).

The following lemmas give the choices available for the involution in the group. As in previous section, a key step is whether the set of symmetric elements, \( G^+ \), is or is not a commutative set. Next lemma, which is the counterpart of lemma 2.2 in the preceeding section, is the most that can be said about it.

**Lemma 3.2.** If \( RG^-\) is anticommutative then either \( G^+ \) is a commutative set or \((R^-)^2 = 0\).
Proof. Assume there are elements $a$ and $b$ in $R^-$ such that $ab \neq 0$ and let $g, h$ be in $G^+$, so $ag$ and $bh$ are skew elements of $RG$. Therefore, they anticommute so $0 = ag \circ bh = ab gh + ba hg = ab(gh - hg)$, the last step because $R^-$, being a subset of $RG^-$, is anticommutative. Since $ab \neq 0$ I get $gh = hg$.

Notwithstanding this last result, the involution in the group can be determined to a much extent under the assumption of anticommutativity of skew-symmetric elements. This is done by Goodaire and Polcino Milies, [12], and I extract their results here for, in this paper, though they are necessary conditions, they are not sufficient.

**Lemma 3.3** (from theorem 2.2 in [12]). *If $RG^-$ is anticommutative and the involution in the group is not the identity, then either the group $G$ is Abelian, has a subgroup $H$ of index 2 in $G$, an element $s$ in $H$ of order 2 and the involution is given by $g^* = g$, if $g \in H$, and $g^* = sg$, if $g \notin H$, or either $G$ is non-Abelian, has a unique nontrivial commutator $s$ (necessarily central and of order 2) and the involution is given by $g^* = g$ or $sg$.*

My task is now to set conditions on the involution in the ring $R$.

**Lemma 3.4.** *If $RG^-$ is anticommutative and the involution in the group is not the identity then the involution in the ring $R$ satisfies*


\begin{enumerate}
\item $2R^- = 0$,
\item $R^-$ is in the centre of $R$ and
\item for any $a, b$ in $R$,
\[ (a \circ b)^* = -a \circ b. \]
\end{enumerate}

**Proof.**

i. Let $a$ be in $R^-$ and $g$ an element in $G$ such that $g^* \neq g$. Then $a$ and $g - g^*$ are skew-symmetric elements so they anticommute: $a \circ (g - g^*) = 0$, which leads to $2ag = 2ag^*$. Since $g^* \neq g$, I must conclude $2a = 0$.

ii. Let $a$ and $g$ be as in (i) and $b$ an arbitrary element of $R$. By lemma 3.1 $g^2$ is a symmetric element so $ag^2$ and $bg - b^* g^*$ are both skew-symmetric elements and, thus: $ag^2 \circ (bg - b^* g^*) = 0$, which leads to $(ab + ba)g^2 = (ab + b^* a)g^2 g^*$. Since $g^* \neq g$, both sides are zero and so $ab = -ba = ba$, because $a = -a$ by the previous item.

iii. Let $a, b$ be arbitrary elements in $R$ and $g$ an element in $G$ such that $g^* \neq g$. The Jordan product $(ag - a^* g^*) \circ (bg - b^* g^*)$ is zero since the factors are skew elements. Its expansion, making use of lemma 3.1, gives

\begin{equation}
(ab + ba + a^* b^* + b^* a^*) g^2 = (ab^* + b^* a + a^* b + ba^*) g g^*.
\end{equation}

Since $g^* \neq g$ both sides must be zero so I get $ab + ba + a^* b^* + b^* a^* = 0$, which can be written as in the lemma.
As a remark, notice that equation (17) gives, among other consequences, the characteristic of the ring is 4 (take \( a = b = 1 \) and remember the char \( \mathbb{R} = 2 \) is explicitly excluded), the set of traces is 2-torsion, \( 2T_{\mathbb{R}} = 0 \), (take \( b = 1 \)), or \( 2\mathbb{R} \subset \mathbb{R}^- \).

There is another necessary condition which needs to be stated for a particular case of the involution in the group \( G \).

**Lemma 3.5.** Let \( G \) be a non-Abelian group with an involution \( * \) such that there are elements \( g, h \in G \) for which \( gh \neq hg \). If \( RG^- \) is anticommutative then the set of traces in \( R \), verifies \((TR)^2 = 0\).

**Proof.** Let \( a \) and \( b \) be arbitrary elements of the ring \( R \), while \( g \) and \( h \) be the elements stated in the lemma. The skew elements \( ag - a^*g^* \) and \( bh - b^*h^* \) have, thus, a trivial Jordan product which leads to

\[
abgh - ab^*gh^* - a^*bg^*h + a^*b^*g^*h^* + b^*ah^*g - ba^*hg^* + b^*a^*h^*g^* = 0. \tag{19}
\]

By lemma 3.3 I know \( g^* = sg \) and \( h^* = sh \), where \( s \) is the commutator \( (g, h) = (h, g) \) which is central, so the terms can be reordered as

\[
(ab + a^*b^* - b^*a - ba^*)gh = (ab^* + a^*b - ba - b^*a^*)hg. \tag{20}
\]

Since \( gh \neq hg \) both sides are zero and, thus, \( ab + a^*b^* = b^*a + ba^* \). Upon substitution of this last expression into equation (17) gives at once \((b + b^*)(a + a^*) = 0\), so the product of traces in \( R \) is trivial. \( \square \)

The necessary conditions hitherto shown turn out to be also sufficient in next theorem which gives a full answer to the problem of the set of skew-symmetric elements in \( RG \) being anticommutative.

**Theorem 3.6.** Let \( G \) be a group with an involution \( * \), \( R \) a ring with characteristic not 2 and \( RG \) the group ring in which an involution, also denoted \( * \), has been defined which is an extension of that in the group \( G \). The set of skew-symmetric elements \( RG^- \) is anticommutative if, and only if, one of the following scenarios is met:

i. \( G \) is an Abelian group and its involution is the identity function. The involution in the ring \( R \) verifies \( R^- \) is an anticommutative set.

ii. \( G \) is an Abelian group with a subgroup \( H \) of index 2 in \( G \), an element \( s \) in \( H \), of order 2, and its involution is given by \( g^* = g \), if \( g \in H \), and \( g^* = sg \), if \( g \notin H \).

The involution in the ring \( R \) verifies \( R^- \) is contained in the centre, is a 2-torsion set and contains every Jordan product of elements of \( R \).

iii. \( G \) is a non-Abelian group with a unique nontrivial commutator \( s \) and the involution in \( G \) is given by \( g^* = g \) or \( sg \).
The involution in the ring $R$ satisfies $R^{-}$ is contained in the centre, it contains every Jordan product of elements of $R$ and, if $G^{+}$ is commutative, then $2R^{-} = (T_{R})^{2} = 0$, while if it is not, then $(R^{-})^{2} = 0$.

Proof. If $G$ is an Abelian group with the identity function as involution then the skew elements in $RG$, according to equation (3), are of the form $\sum_{g \in G} a_{g}g$, where the coefficients $a_{g}$ are in $R^{-}$ and, thus, their anticommutativity is assured if $a \circ b = 0$ for any $a$ and $b$ in $R^{-}$. Sufficiency of the item (i) is then proved.

In the remaining cases the condition $(a \circ b)^{*} + a \circ b = 0$, for any $a$ and $b$ in the ring $R$ is found, and this implies the characteristic of the ring is 4.

If the involution in the group is not the identity, but the group is still Abelian, lemmas 3.3 and 3.4 say the conditions on (ii) are necessary. To see they are also sufficient I must check the Jordan product in the set $R \cup S$ is trivial. Jordan products among elements of $R$ and between an element of $R$ and an element of $S$ are zero if $ag \circ bh$ is zero for $a \in R^{*}$, $g \in G^{+}$ while $b$ is an arbitrary element of the ring $R$ and $h$ is arbitrary in the group $G$. But this Jordan product is $ag \circ bh = abgh + ba hg = (ab + ba) gh = 0$ since $a$, which is in the centre of the ring, equals $-a$. The third Jordan product is of the form $(ag-a^{*}g^{*}) \circ (bh-b^{*}h^{*})$, where $a$ and $b$ are arbitrary elements of $R$ while $g$ and $h$ are in $G \setminus G^{+}$. Its expansion, taking into account the hypothesis on the group involution, gives

$$((a \circ b)^{*} + a \circ b) gh - ((a \circ b)^{*} + a \circ b^{*}) sgh,$$

where each coefficient is zero because Jordan products in $R$ are skew-symmetric.

If the group is non-Abelian, but the subset $G^{+}$ is commutative, we are in front of a SLC group as discussed above. Then, lemmas 3.3, 3.4 and 3.5 say the conditions in the first part of (iii) are necessary. For sufficiency, since $G^{+}$ is the centre of the group, the arguments given in the previous paragraph to prove the Jordan product in $R$ is trivial as well as between an element of $R$ and an element of $S$ apply here too. I focus, then, on the set $S$: the Jordan product is $(ag-a^{*}g^{*}) \circ (bh-b^{*}h^{*})$ for $a, b \in R, g, h \in G \setminus G^{+}$. If $gh = hg$, then the proof of previous paragraph also applies here. If $gh \neq hg$ then this Jordan product expands as

$$(ab + a^{*}b^{*} - b^{*}a - ba^{*}) gh + (ba + b^{*}a^{*} - ab^{*} - a^{*}b) hg.$$

But, from the condition on traces, $(b + b^{*})(a + a^{*}) = 0$ and I get $-b^{*}a - ba^{*} = ba + b^{*}a^{*}$. Upon substitution in the coefficient of the first term, this coefficient is written as $ab + a^{*}b^{*} + ba + b^{*}a^{*} = (a \circ b)^{*} + a \circ b$ which is zero by hypothesis. The same applies to the other coefficient, so the Jordan product is zero.

Finally, if the group is not Abelian and the subset $G^{+}$ is not commutative, then lemma 3.2 says that $(R^{-})^{2} = 0$ is an additional necessary condition to those of lemmas 3.3 and 3.4 so this is the second part of (iii). Sufficiency follows. If $a, b$ are in $R^{-}$ and $g, h$ in $G^{+}$ then the Jordan product in $R$ is trivial because $ab = ba = 0$. The second Jordan product is $ag \circ (bh - b^{*}h^{*}) = ab gh - ab^{*} gh^{*} + ba hg - b^{*}a h^{*} g$. If $gh = hg$ this amounts to $2ab gh - 2ab^{*} gh^{*}$.
because $a$ is in the centre of $R$. Since $2a = 0$, because both, 2 and $a$, are skew elements, that is zero. Now, if $gh \neq hg$, then $gh^* = hg$ and $h^*g = gh$, so the Jordan product is $a(b − b^*)(gh + hg)$ which is zero because $a$ and $b − b^*$ are both in $R^−$ where the product is trivial. That the Jordan product in $S$ is trivial is proved in exactly the same manner as in previous paragraph once I show the condition $(T_R)^2 = 0$ is concealed within the hypothesis of the present case. For any $a$ in $R$, $2a = 1 \circ a$, being a Jordan product, is a skew element in $R$. From this I deduce $2(a + a^*) = 0$, this is, traces are 2-torsion elements and, thus, skew-symmetric, so the product of traces is trivial.

References


